Mid term
Fall (2009-2010)

1) (10 pts)Find the length of the curve:

$$x = 9\cos^3 t \text{ and } y = 9\sin^3 t \text{ when } 0 \le t \le \frac{\pi}{4}$$

$$\frac{dx}{dt} = -27\cos^2 t \cdot \sin t$$

$$\frac{dy}{dt} = 27\sin^2 t \cdot \cos t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (27)^2\left[\cos^4 t \sin^2 t + \sin^4 t \cdot \cos^4 t\right]$$

$$= (27)^2 \cos^2 t \sin^2 t\left[\underbrace{\cos^2 t + \sin^2 t}_{1}\right]$$

$$= \left(27\cos t \cdot \sin t\right)^2$$

$$L = \int_0^{\frac{\pi}{4}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 27\int_0^{\frac{\pi}{4}} \sin t \cdot \cos t \, dt$$

$$L = 27\left(\frac{\sin^2 t}{2}\right)\Bigg|_0^{\frac{\pi}{4}}$$

$$L = \frac{27}{2}\left(\frac{1}{2}\right) = \frac{27}{4} \quad (\text{u.L})$$

2) (10 pts) Find the cosine of the angle between the 2 vectors:

$\vec{u} = 3\vec{i} + 4\vec{j}$ and $\vec{v} = 2\vec{i} - \frac{\vec{j}}{2}$

$\vec{u} \cdot \vec{v} = 6 - 2 = 4$

$\|\vec{u}\| = \sqrt{9+16} = 5$

$\|\vec{v}\| = \sqrt{4 + \frac{1}{4}} = \frac{\sqrt{17}}{2}$

$\cos(\vec{u}, \vec{v}) = \dfrac{4}{5 \times \frac{\sqrt{17}}{2}} = \dfrac{8}{5\sqrt{17}}$

3) (15 pts – 5 pts each)Solve the following

a) $I=\int \frac{\cos(3\ln x)}{x}dx$

let $u = 3\ln x$

$du = \frac{3}{x}dx$

$I = \frac{1}{3}\int \cos u \, du$

$I = \frac{1}{3}\sin u + C$

$I = \frac{1}{3}\sin(3\ln x) + C$

b) $\int \frac{dx}{2+e^x} = \int \frac{dx}{e^x(2e^{-x}+1)}$

$= \int \frac{e^{-x}\,dx}{2e^{-x}+1}$ let $u = 2e^{-x}+1$

$du = -2e^{-x}dx$

$= -\frac{1}{2}\int \frac{du}{u}$

$= -\frac{1}{2}\ln|u| + C$

$= -\frac{1}{2}\ln(2e^{-x}+1) + C$

c) $I = \int \frac{\left(1+\sqrt{x}\right)^{\frac{1}{3}}}{\sqrt{x}}\,dx$

let $u = 1 + \sqrt{x}$

$du = \frac{dx}{2\sqrt{x}}$

$$I = 2\int u^{\frac{1}{3}}\,du$$

$$I = \cancel{2}\left(\frac{u^{\frac{1}{3}+1}}{\frac{4}{3}}\right) + c$$

$$I = \frac{3}{2}\left(1+\sqrt{x}\right)^{\frac{4}{3}} + C$$

4) (10 pts)Evaluate $\frac{dy}{dx}$ using logarithmic differentiation

$$y = \frac{\left(x^2+3x\right)\cos\left(x^2\right)\tan^{-1}(3x)}{\sqrt[3]{(x+2)^2(x-1)^5}\,.\left[e^{5x+4}\right]}$$

$$\ln y = \ln\left(x^2+3x\right) + \ln\left(\cos(x^2)\right) + \ln\left(\tan^{-1}(3x)\right) - \frac{1}{3}\left[2\ln(x+2) + 5\ln(x-1)\right] - (5x+4)$$

$$\frac{y'}{y} = \frac{2x+3}{x^2+3x} - \frac{2x\sin(x^2)}{\cos(x^2)} + \frac{3}{(1+9x^2)\tan^{-1}(3x)} - \frac{2}{3(x+2)} - \frac{5}{3(x-1)} -$$

$$\therefore \quad y' = y\left[\frac{2x+3}{x^2+3x} - \frac{2x\sin(x^2)}{\cos(x^2)} + \frac{3}{(1+9x^2)\tan^{-1}(3x)} - \frac{2}{3(x+2)} - \frac{5}{3(x-1)} - 5\right]$$

5) (15 pts – 5 pts each)Find $\frac{dy}{dx}$ of the following functions

a) $e^{x^2+y} = \cos^{-1}(xy)$

$$\left(2x+\frac{dy}{dx}\right)e^{x^2+y} = \left(y+x\frac{dy}{dx}\right)\left[\frac{-1}{\sqrt{1-(xy)^2}}\right]$$

$$\frac{dy}{dx}\left(e^{x^2+y} + \frac{x}{\sqrt{1-(xy)^2}}\right) = -2x\,e^{x^2+y} - \frac{y}{\sqrt{1-(xy)^2}}$$

$$\therefore \quad \frac{dy}{dx} = \left[-2x\,e^{x^2+y} - \frac{y}{\sqrt{1-(xy)^2}}\right] \left[\frac{1}{e^{x^2+y} + \frac{x}{\sqrt{1-(xy)^2}}}\right]$$

b) $y = \ln\left(\frac{5x^2\, e^{-3x}}{\ln x + 4}\right)$

$$y = \underbrace{\ln(5x^2)}_{\ln 5 + 2\ln x} - 3x - \ln(\ln x + 4)$$

$$\frac{dy}{dx} = \frac{2}{x} - 3 - \frac{1}{x(\ln x + 4)}$$

$$\sqrt[3]{2xy} = (2xy)^{\frac{1}{3}}$$

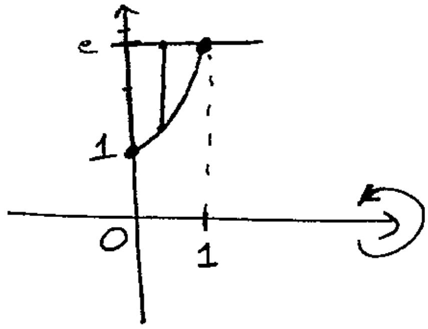
$$u = \qquad u' = \frac{1}{3}(2y + 2xy')(2xy)^{-\frac{2}{3}}$$

c) $5^{\sqrt[3]{2xy}} = 3y^2 - 2$

$$\frac{2}{3}\left(y + x\frac{dy}{dx}\right)\left(5^{\sqrt[3]{2xy}}\ln 5\right)\cdot(2xy)^{-\frac{2}{3}} = 6y\frac{dy}{dx}$$

$$\left[2x(2xy)^{-\frac{2}{3}}\left(5^{\sqrt[3]{2xy}}\ln 5\right) - 6y\right]\frac{dy}{dx} = -2y(2xy)^{-2/3}\left(5^{\sqrt[3]{2xy}}\ln 5\right)$$

6) (10 pts)Find the volume generated by revolving the region bounded by the curves $y = e^x$, $y = e$ and x=0 about the x-axis.



$$V = \pi \int_0^1 \left(e^2 - e^{2x}\right) dx$$

$$V = \pi \left[e^2 x - \frac{1}{2} e^{2x} \right]_0^1$$

$$V = \pi \left[e^2 - \frac{1}{2} e^2 + \frac{1}{2} \right]$$

$$V = \pi \left[\frac{1}{2} e^2 + \frac{1}{2} \right]$$

$$V = \frac{\pi}{2} \left(e^2 + 1 \right) \quad (u.v)$$

7) (10 pts)Find the length of the curve: $y = x^2 - \frac{\ln x}{8}$ when $1 \le x \le 2$

$$y' = 2x - \frac{1/x}{8}$$

$$y'^2 = 4x^2 - \frac{1}{2x} + \frac{1}{64x^2}$$

$$1 + y'^2 = 4x^2 + \frac{1}{2x} + \frac{1}{64x^2}$$

$$\therefore\ 1 + y'^2 = \left(2x + \frac{1}{8x}\right)^2$$

$$L = \int_1^2 \sqrt{\left(2x + \frac{1}{8x}\right)^2}\, dx$$

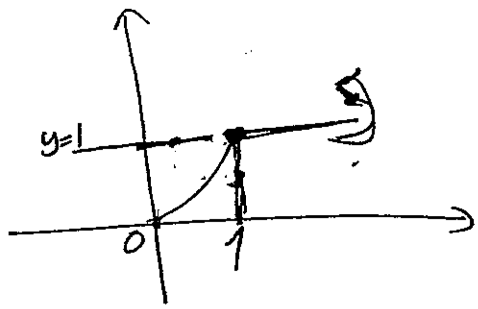
$$L = \int_1^2 \left(2x + \frac{1}{8x}\right) dx$$

$$L = x^2 + \frac{\ln x}{8} \Big]_1^2$$

$$L = 3 + \frac{\ln 2}{8}$$ (u.L)

8) (20 pts – 10 pts each)

a) Find the volume of the solid generated by rotating the region $0 \le y \le x^3$ and $0 \le x \le 1$ about the line $y = 1$



$y = x^3 \Rightarrow x = y^{1/3}$

$$V = 2\pi \int_0^1 (1 - y^{\frac{1}{3}})(1 - y)\, dy$$

$$V = 2\pi \int_0^1 \left(1 - y - y^{\frac{1}{3}} + y^{\frac{4}{3}}\right) dy$$

$$V = 2\pi \left[y - \frac{y^2}{2} - \frac{3y^{\frac{4}{3}}}{4} + \frac{3y^{\frac{7}{3}}}{7} \right]_0^1$$

$$V = 2\pi \left[1 - \frac{1}{2} - \frac{3}{4} + \frac{3}{7} \right]$$

(u.v)

$$V = 2\pi \left(\frac{5}{28} \right) = \frac{5\pi}{14}$$

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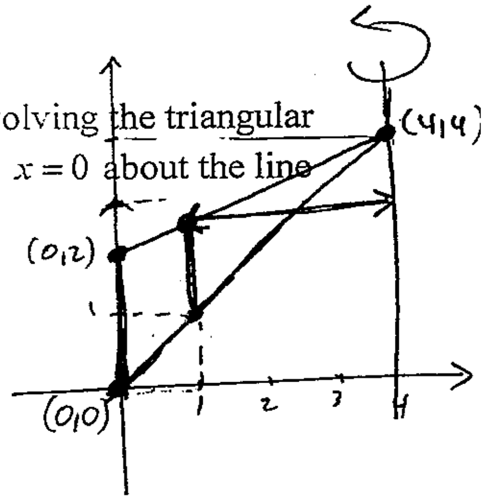
or $$V = \int_0^1 \pi \left(1^2 - (1 - x^3)^2 \right) dx$$

$$= \int_0^1 \pi \left(2x^3 - x^6 \right) dx$$

$$= \pi \left[\frac{2x^4}{4} - \frac{x^7}{7} \right]_0^1 = \frac{5\pi}{14}$$

b) Compute the volume of the solid generated by revolving the triangular region bounded by the lines $2y = x + 4$, $y = x$ and $x = 0$ about the line $x = 4$ using the Shell Method



$$2y = x + 4$$
$$y = \frac{x}{2} + 2$$

$$\text{height} = \frac{x}{2} + 2 - x = -\frac{x}{2} + 2$$

$$\text{radius} = 4 - x$$

$$V = 2\pi \int_0^4 \left(-\frac{x}{2} + 2\right)(4 - x)\, dx$$

$$V = 2\pi \int_0^4 \left(-4x + \frac{x^2}{2} + 8\right) dx$$

$$V = 2\pi \left[-2x^2 + \frac{x^3}{6} + 8x\right]_0^4$$

$$V = 2\pi \left[-\cancel{32} + \frac{32}{3} + \cancel{32}\right]$$

$$V = \frac{64\pi}{3} \quad (u.v)$$

BONUS:(10 PTS)

Solve the following integrals

a) $\int_{1}^{\sqrt[4]{e}} \frac{\tan^2 \pi \ln^2 x}{x} dx$

$= \tan^2 \pi \left[\frac{\ln^3 x}{3} \right]_1^{\sqrt[4]{e}}$

$= \tan^2 \pi \left[\frac{1}{3} \left(\ln^3 e^{\frac{1}{4}} - 0 \right) \right]$

$= \frac{1}{12} \tan^2 \pi$

$= 0$ because

$\tan \pi = 0$

b) $\int \frac{\sec^2 x}{1+\tan^2 x} dx$

$1 + \tan^2 x = \sec^2 x$

$= \int dx$

$= x + C$.